2D CFT and the Ising Model

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December 27, 2017

Abstract

In this lecture, we review the results of Appendix E from the seminal paper by Belavin, Polyakov, and Zamolodchikov on the critical scaling exponents of the 2D Ising model by studying the conformal field theory of a free massless fermion system.

1 Review of 2D CFT Basics

Recall that in the complex plane, we have that the conformal transformations are precisely holomorphic functions $z \to w(z), \bar{z} \to \bar{w}(\bar{z})$. In a local neighborhood, we have a taylor series expansion in terms of the generators:

$$l_n = -z^{n+1}\partial_z, \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{1}$$

with relations

$$[l_m, l_n] = (m-n)\zeta_{m+n} \tag{2}$$

this is the **Witt Algebra**.

Definition 1.1 (Stress Tensor)**.** For an action S, we define the stress-energy tensor as:

$$T_{\mu\nu} := \frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \tag{3}$$

Tracelessness gives that we can diagonalize T just as

$$\begin{pmatrix} T_{zz} & 0 \\ 0 & T_{\bar{z}\bar{z}} \end{pmatrix} \tag{4}$$

And $T(z) := T_{zz}(z)$

Writing

$$T(z) = \sum_n \frac{L_n}{z^{n+2}} \Rightarrow L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \tag{5}$$

gives element L_n that form a central extension of the Witt algebra known as the **Virasoro algebra**.

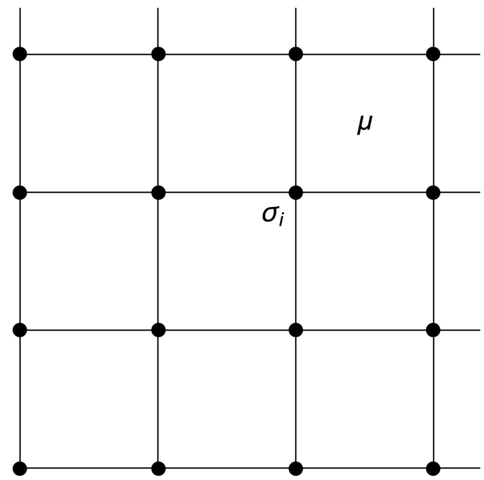
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{6}$$

This parameter c is called the **central charge**.

2 2D Ising Model and Majorana Fermions

We characterize the classical 2D ising model by a square lattice, holding at each vertex i a spin σ_i that is either $+1/2$ or $-1/2$. This model is exactly soluble by reduction to a so-called 1-D quantum ising model, and then using the Jordan-Wigner transform to write it as a free-fermion on the line.

Conformal field theory is a very powerful tool in studying such statistical models at critical temperature, and its use in working with the 2D ising model goes back to the early 70s. We wish to use a different approach to motivate the free Majorana fermion on the complex plane, arising from the 2D ising model.



In addition to the spin σ (referred to commonly as the **order parameter**), we can consider a "Fourier transform" of this model, and associate to each point in the **dual lattice** (i.e. the lattice of squares in between the vertices, connected across edges).

For a given spin-distrubution σ on the sites, there is a corresponding **disorder parameter**, μ, on the dual lattice. This naming scheme comes from a deep property of the Ising model known as **Kramers-Wannier duality**, that the statistics of σ at low/high temperature are equivalent to the statistics of μ at a corresponding high/low temperature, respectively. That is, at low temperatures where the spins align and σ is "ordered", we get that μ is disordered, but at high temperatures, the disorder of the spins gives order in the corresponding μ statistics. This idea is closely related to the Heisenberg principle, of how a sharply peaked distribution gives rise to a frequency distribution of very large support and vice/versa.

The 2D Ising model, in the continuous limit, is then equivalent to the theory of free Majorana fermions, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\psi\frac{\partial}{\partial\bar{z}}\bar{\psi} + \frac{1}{2}\bar{\psi}\frac{\partial}{\partial\bar{z}}\psi + m\bar{\psi}\psi \tag{7}$$

with m proportional to $T - T_c$. Note that because this fermions has no charge, the field is real valued: $\psi^* = \psi$, so $\bar{\psi}$ is not the same as the conjugate of ψ, but is rather a second, independent component.

We care about at the critical temperature $T = T_C$, where the mass becomes zero and the equations of motion with $m = 0$ give:

$$\frac{\partial}{\partial\bar{z}}\psi = 0, \ \frac{\partial}{\partial z}\bar{\psi} = 0. \tag{8}$$

Namely that ψ is holomorphic so $\psi = \psi(z)$ and $\bar{\psi} = \bar{\psi}(\bar{z})$ is anti-holomorphic. Then the stress-energy tensor can be obtained directly

$$\begin{aligned} T(z) &= -\frac{1}{2}\psi(z)\frac{\partial}{\partial z}\psi(z) \\ T(\bar{z}) &= -\frac{1}{2}\bar{\psi}(\bar{z})\frac{\partial}{\partial \bar{z}}\bar{\psi}(\bar{z}) \end{aligned} \tag{9}$$

From $T(z)$, we obtain a Virasoro algebra with central charge $1/2$. Moreover, we can directly invert the kinetic part of the Lagrangian to obtain equations for the propogators:

$$\begin{aligned} \langle \psi(z)\psi(w) \rangle &= \frac{1}{z-w} \\ \left\langle \bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) \right\rangle &= \frac{1}{\bar{z}-\bar{w}} \end{aligned} \tag{10}$$

We know that two-point functions of a primary field with scaling dimensions $(\Delta, \bar{\Delta})$ behave as

$$\langle \phi(z)\phi(w) \rangle \propto \frac{1}{(z-w)^{2\Delta}(\bar{z}-\bar{w})^{2\bar{\Delta}}} \tag{11}$$

and in fact $\psi, \bar{\psi}$ are primary with scaling dimensions

$$\begin{aligned} (\Delta_\psi, \bar{\Delta}_\psi) &= (1/2, 0) \\ (\Delta_{\bar{\psi}}, \bar{\Delta}_{\bar{\psi}}) &= (0, 1/2) \end{aligned} \tag{12}$$

So in fact the field transforms as

$$\begin{aligned} \psi(z) &\to \left(\frac{\partial \zeta}{\partial z}\right)^{1/2} \psi(\zeta) \\ \bar{\psi}(\bar{z}) &\to \left(\frac{\partial \zeta}{\partial z}\right)^{1/2} \bar{\psi}(\bar{z}) \end{aligned} \tag{13}$$

Proposition 2.1. *The set of fields $(1, \psi, \bar{\psi}, \psi\bar{\psi})$ forms a **complete set** in the sense that these fields, together with their descendants, are closed under operator product expansions.*

Their corresponding scaling dimensions are

$$1 \to (0,0),\ \psi \to (1/2, 0),\ \bar{\psi} \to (0, 1/2),\ \psi\bar{\psi} \to (1/2, 1/2) \tag{14}$$

3 Differential Equations for Primary Correlators

The operator product expansion of T with $\psi(z)$ gives

$$T(\zeta)\psi(z) = \left[\frac{1}{2}\frac{1}{(\zeta - z)^2} + \frac{1}{\zeta - z}\partial_z + \frac{3}{4}\frac{\partial^2}{\partial^2 z}\right]\psi(z) \tag{15}$$

Further recall the **Ward Identity** applied to $\langle TX \rangle$, where $X = \phi(w_1)\dots\phi(w_n)$ is a product of *primary operators.*

$$\langle TX \rangle = \sum_{i=1}^{n} \left(\frac{1}{z-w_i}\partial_{w_i} + \frac{\Delta_i}{(z-w_i)^2}\right)\langle X \rangle + reg. \tag{16}$$

where *reg.* stands for higher-order holomorphic terms.

Now the same equation holds for the product $\psi(z)\phi(w_1)\ldots\phi(w_n)$ of primary operators, giving for $\langle T\psi(z)\phi(w_1)\ldots\phi(w_n)\rangle$ the result:

$$\left[\frac{1}{\zeta - z}\partial_z + \frac{\frac{1}{2}}{(\zeta - z)^2} + \sum_{i=1}^{n}\left(\frac{1}{\zeta - w_i}\partial_{w_i} + \frac{\Delta_i}{(\zeta - w_i)^2}\right)\right]\langle X\rangle + reg. \tag{17}$$

On the other hand from OPE of $T\psi$, we get that this is equal to

$$\left[\frac{1}{2}\frac{1}{(\zeta - z)^2} + \frac{1}{\zeta - z}\partial_z + \frac{3}{4}\frac{\partial^2}{\partial^2 z}\right]\langle X\rangle . \tag{18}$$

Together, these give a differential equation for $\langle X\rangle$:

$$\left[\frac{3}{4}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n}\left(\frac{1}{\zeta - w_i}\partial_{w_i} + \frac{\Delta_i}{(\zeta - w_i)^2}\right)\right]\langle X\rangle = 0 \tag{19}$$

We can use this, now, as a powerful tool in finding correlation functions.

4 Ising Correlators of σ and μ

The original lattice order parameter σ, representing the spin at site i, in the continuum limit becomes an operator $\sigma(z, \bar{z})$, corresponding to the spin density at a point. Similarly μ becomes a continuous variable $\mu(z, \bar{z})$. We expect $\Delta_\sigma = \Delta_\mu$ because of Ising duality.

Moreover, from the statistical model, we know that the correlator

$$\langle \psi(z)\sigma(\xi_1)\ldots\sigma(\xi_{2N-1})\mu(\xi_{2N})\ldots\mu(\xi_{2M})\rangle \tag{20}$$

is a multi-valued function in z, picking up a $-$ sign if z circles a ξ_i. This corresponds to an orientation reversal. For this reason, we get square-root behaviour in the OPE:

$$\begin{aligned}\psi(\xi)\sigma(z, \bar{z}) &= (\xi - z)^{-1/2}(\mu(z, \bar{z}) + reg.)\\ \psi(\xi)\mu(z, \bar{z}) &= (\xi - z)^{-1/2}(\sigma(z, \bar{z}) + reg.)\end{aligned} \tag{21}$$

In fact σ, μ are primary. With these OPEs and the differential equation from before for $\langle \psi X\rangle$, we get

$$\Delta_\sigma = \Delta_\mu = 1/16 = \bar{\Delta}_\sigma = \bar{\Delta}_\mu \tag{22}$$

This moreover gives that 2-point functions $\langle\sigma\sigma\rangle$ and $\langle\mu\mu\rangle$ scale as

$$\frac{1}{(z - w)^{1/4}},$$

which is exactly the scaling behaviour of the Ising correlators at critical temperature.

Now, as before, using the same trick of expanding $\langle TX\rangle$ by Ward identity, we consider

$$\langle X\rangle = \langle\sigma(\xi_1)\ldots\sigma(\xi_{2N})\mu(\xi_{2N+1})\ldots\mu(\xi_{2M})\rangle \tag{23}$$

Together with using the fact that σ, μ are in fact degenerate operators, we get one differential equation for each of the coordinates ξ_i:

$$\left[\frac{3}{4}\frac{\partial^2}{\partial z_i^2} - \sum_{j\neq i}^{2M}\left(\frac{1}{z_i - z_j}\partial_{z_j} + \frac{\frac{1}{16}}{(z_i - z_j)^2}\right)\right]\langle X\rangle = 0 \tag{24}$$

We shall now apply this to calculate 4-pt functions.

5 4-point Correlators

For the correlator $\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle$, we have a full description in terms of the cross-ratios $x, \bar{x}$ that the correlation function should look like:

$$\frac{Y(x,\bar{x})}{(z_{13}z_{24}\bar{z}_{13}\bar{z}_{24})^{1/8}} \tag{25}$$

We obtain 2 ODEs, on for each of $x, \bar{x}$. Together they combine to give, in terms of x:

$$\left[x(1-x)\frac{\partial^2}{\partial x^2} + (\frac{1}{2}-x)\frac{\partial}{\partial x} + \frac{1}{16}\right]u(x,\bar{x}) = 0 \tag{26}$$

where

$$u(x,\bar{x}) = [x\bar{x}(1-x)(1-\bar{x})]^{1/8}\,Y(x,\bar{x}) \tag{27}$$

is Y, rescaled appropriately. Through change of variables, $x = \sin^2(\theta), \bar{x} = \sin^2(\bar{\theta})$ we get an expression for u as

$$\begin{aligned} u(\theta,\bar{\theta}) =& u_{11}\cos(\theta/2)\cos\big(\bar{\theta}/2\big) + u_{12}\cos(\theta/2)\sin\big(\bar{\theta}/2\big) \\ & u_{21}\sin(\theta/2)\cos\big(\bar{\theta}/2\big) + u_{22}\sin(\theta/2)\sin\big(\bar{\theta}/2\big) \end{aligned} \tag{28}$$

Note that unlike something like $\langle\psi\sigma\sigma\sigma\rangle$, the four-point function $\langle\sigma\sigma\sigma\sigma\rangle$ is well-defined and not double-valued. This condition immediately requires $u_{21} = u_{12} = 0$. Moreover symmetry under $\theta \leftrightarrow -\theta$ gives $u_{11} = u_{22}$, and finally the normalization convention for the two-point function

$$\langle\sigma(z,\bar{z})\sigma(w,\bar{w})\rangle = \frac{1}{(z-w)^{1/4}} \tag{29}$$

gives $u_{11} = u_{22} = 1$ so that all together we get a 4-point function

$$\langle\sigma\sigma\sigma\sigma\rangle = \frac{\cos(\theta/2)\cos\big(\bar{\theta}/2\big) + \sin(\theta/2)\sin\big(\bar{\theta}/2\big)}{[x\bar{x}(1-x)(1-\bar{x})]^{1/8}} \tag{30}$$

and in fact this can easily be expressed in terms of the **Virasoro conformal blocks** $\mathcal{F}(\Delta_1, \Delta_2, x)$ as

$$\mathcal{F}(1/16, 0, x)\bar{\mathcal{F}}(1/16, 0, \bar{x}) + \mathcal{F}(1/16, 1/2, x)\bar{\mathcal{F}}(1/16, 1/2, \bar{x}) \tag{31}$$

And we can recognize that the first product of blocks corresponds to the identity operator (with scaling dimensions $(0,0)$) appearing in the OPE, while the second corresponds to the energy density ε (scaling dimension $(1/2, 1/2)$) appearing.

Similarly, for $\langle\sigma\mu\sigma\mu\rangle$ we get

$$\mathcal{F}(1/16, 0, x)\bar{\mathcal{F}}(1/16, 1/2, \bar{x}) + \mathcal{F}(1/16, 1/2, x)\bar{\mathcal{F}}(1/16, 0, \bar{x}) \tag{32}$$

Corresponding to the operators ψ, $\bar{\psi}$ appearing the OPE of $\sigma\mu$. Indeed, this gives us that

$$\sigma\mu = z^{3/8}\bar{z}^{-1/8}(\psi(z) + reg.) + z^{-1/8}\bar{z}^{3/8}(\psi(\bar{z}) + reg.) \tag{33}$$

Which confirms the idea of $\psi, \bar{\psi}$ being regularized versions of the classical Ising quantities $\sigma\mu$.